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Scaling for domain growth in the Ising model with competing dynamics

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We study the domain growth of the one-dimensional kinetic Ising model under the competing influence of Glauber dynamics at temperature T and Kawasaki dynamics with a configuration-independent rate. The scaling of the structure factor is shown to have the form for nonconserved dynamics with the corrections arising from the spin-exchange process, i.e., $S(k,t)=Lg_0(kL,t/\tau)+g_1(kL,t/\tau)+\cdots$, and the corresponding scaling functions are calculated analytically. A correction to the Porod law at zero temperature is also given.
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The kinetic Ising models have been widely studied to understand the far-from-equilibrium phenomena such as the nonconserved Glauber model [1] and the conserved Kawasaki model [2]. In recent years, the systems with more complex mechanisms, e.g., the Ising models with competing Glauber and Kawasaki dynamics [3,4], have been introduced to investigate the basic questions of nonequilibrium phase transitions and critical phenomena. For these models with competing dynamics, much of the interest has been focused on the properties of nonequilibrium steady state, including the nonequilibrium phase diagrams [4–6] and the critical exponents [6,7]. However, the problem of domain growth, which describes the behavior of the system quenched into an ordered phase from a high-temperature initial state and is important in understanding the dynamics of nonequilibrium processes, has received relatively little attention.

Our interest here is in the study of domain growth for the Ising system with competing dynamics. It is well known [8,9] that in the late stage regime, the nonequilibrium process of domain growth exhibits dynamic scaling behaviors and the scaling forms of the equal-time pair correlation function $C(r,t)$ and its Fourier transformation, the structure factor $S(k,t)$, are given by $C(r,t)=f(r/L(t))$ and $S(k,t)=L(t)^d g(kL(t))$, respectively, where $L(t)$ is the single length scale characterizing the domain structure and d is the spatial dimensionality. The corrections to the above scaling arising from the nonscaling initial condition were determined recently at zero temperature [10]. However, the direct demonstration of scaling and the exact calculation of the scaling functions are still lacking except in some simple models [10–13]. Thus it is of interest to give a direct and exact calculation of scaling behaviors for the system with more complex dynamics.

The system considered here is the one-dimensional Ising model evolving by a combination of the spin-flip Glauber process at temperature T and the spin-exchange Kawasaki process with a configuration-independent rate that occurs as if the system were at infinite temperature [3,14]. In this work we study the domain growth of this soluble model following a sudden quench from an initial high temperature to a final temperature T and calculate the corresponding scaling properties analytically.

The Hamiltonian for this one-dimensional system is given by

$$\mathcal{H}=-J\sum_{i=1}^{N}\sigma_i\sigma_{i+1}, \tag{1}$$

where $\sigma_i=\pm 1$ is the Ising spin variable. The configuration of the system evolves with time via the combination of Glauber dynamics with the spin-flip rate $W_i^{(1)}=(2\tau_1)^{-1}[1-\gamma\sigma_i(\sigma_{i-1}+\sigma_{i+1})/2]$, where $\gamma=\tanh 2K$ with $K=J/k_BT$, and Kawasaki dynamics with the configuration-independent spin-exchange rate $W_{ii+1}^{(2)}=(2\tau_2)^{-1}(1-\sigma_i\sigma_{i+1})$. The equations for the expectation value of the spin and the equal-time pair correlation function have been derived [14] and the latter is written as

$$\begin{aligned}\tau' dC(r,t)/dt&=-2C(r,t)+\gamma'[C(r-1,t)+C(r+1,t)] \quad (r\geqslant 2),\\ \tau' dC(1,t)/dt&=-\frac{2+\kappa}{1+\kappa}C(1,t)+\gamma' C(2,t)+\gamma'-\frac{\kappa}{1+\kappa},\\ C(0,t)&=1\end{aligned} \tag{2}$$

provided the initial probability distribution is translationally invariant, where the pair correlation function $C(r,t)=\langle\sigma_i(t)\sigma_{i+r}(t)\rangle$, $\kappa=2\tau_1/\tau_2$, $\tau'=\tau_1/(1+\kappa)$, and $\gamma'=(\gamma+\kappa)/(1+\kappa)$.

From Eq. (2) the scaling results for domain growth can be calculated directly following the method given by Bray [12]. By Fourier transforming in space, the equation for the structure factor $S(k,t)$ is obtained,

$$\begin{aligned}\tau' dS(k,t)/dt&=-\gamma_k S(k,t)+A(t)-\frac{2\kappa}{1+\kappa}n(t)\cos k,\\ \frac{1}{N}\sum_k S(k,t)&=1,\end{aligned} \tag{3}$$

where $\gamma_k=2(1-\gamma'\cos k)$, $A(t)=\Sigma_k\gamma_k S(k,t)/N$, and $n(t)=[1-C(1,t)]/2$ denotes the average wall density. This equation can be solved by a Laplace transformation in time via $\overline{S}(k,s)=\int_0^\infty dt\,\exp(-st)S(k,t)$. In the scaling regime of late times after the quench, the characteristic domain size $L(t)$ is known to obey the growth law $L(t)\sim t^x$, where the growth exponent x is 1/2 and 1/3 for systems with nonconserved and conserved scalar order parameters, respectively [8,9]. Its inverse, the average wall density, then satisfies $n(t)=bt^{-x}/2=bt^{\nu-1}/2$ with $0<\nu<1$, as well as a certain coefficient b, and the corresponding Laplace transform is $n(s)=b\Gamma(\nu)s^{-\nu}/2$. Here we assume that the above scaling forms of the domain size and the wall density remain for this model with competing dynamics, which will be justified *a posteriori*. Thus, on the condition that the initial state contains no long-range order, in the scaling limit ($s\to 0$, $k\to 0$ with $s/k^{1/x}$ arbitrary) we have

$$\overline{S}(k,s)=\left\{(\tau' s+\kappa'^2)^{1/2}\left[2s^{-1}-\frac{\kappa}{1+\kappa}\left(1+\frac{\kappa'^2}{2}\right)b\Gamma(\nu)s^{-\nu}\right]+\frac{\kappa}{1+\kappa}\,\frac{k^2+\kappa'^2}{2}\,b\Gamma(\nu)s^{-\nu}\right\}(\tau' s+k^2+\kappa'^2)^{-1} \quad (4)$$

when domains grow at small but nonzero temperature T. Here $\kappa'=\xi^{-1}\simeq 2\exp(-2K)/\sqrt{1+\kappa}$, where ξ is the correlation length for this system with competing dynamics [14].

Using the inverse Laplace transform on Eq. (4) yields

$$\begin{aligned}S(k,t)={}&2\left(\frac{t}{\pi\tau'}\right)^{1/2}\frac{1}{k^2+\kappa'^2}\int_0^1 dy\,y^{-1/2}\left[\kappa'^2\exp\left(-\frac{\kappa'^2 t}{\tau'}y\right)+k^2\exp\left(-\frac{k^2+\kappa'^2}{\tau'}t+\frac{k^2 t}{\tau'}y\right)\right]\\&+\frac{\kappa b}{1+\kappa}\left\{\left(\frac{1}{\pi\tau'}\right)^{1/2}\left(1+\frac{\kappa'^2}{2}\right)t^{\nu-1/2}\left[-B\left(\frac{1}{2},\nu\right)\,{}_1F_1\left(\nu;\frac{1}{2}+\nu;\frac{\kappa'^2 t}{\tau'}\right)\exp\left(-\frac{\kappa'^2 t}{\tau'}\right)\right.\right.\\&\left.+\frac{1}{\nu}\,\frac{k^2 t}{\tau'}\int_0^1 dy\,y^{-1/2}(1-y)^{\nu}\,{}_1F_1\left(\nu;1+\nu;\frac{k^2+\kappa'^2}{\tau'}t(1-y)\right)\exp\left(-\frac{k^2+\kappa'^2}{\tau'}t+\frac{k^2 t}{\tau'}y\right)\right]\\&\left.+\frac{k^2+\kappa'^2}{2\tau'\nu}t^{\nu}\,{}_1F_1\left(\nu;1+\nu;\frac{k^2+\kappa'^2}{\tau'}t\right)\exp\left(-\frac{k^2+\kappa'^2}{\tau'}t\right)\right\},\end{aligned} \quad (5)$$

where $B(x,y)$ is the beta function and ${}_1F_1(\alpha;\beta;z)$ is the degenerate hypergeometric function [15]. It is shown from Eq. (5) that a scaling solution requires $L(t)\sim t^{1/2}$, i.e., $\nu=1/2$, and then the equal-time structure factor has the scaling form

$$S(k,t)=t^{1/2}g_0(k^2t,\kappa'^2t)+g_1(k^2t,\kappa'^2t)+t^{-1/2}g_2(k^2t,\kappa'^2t)+t^{-1}g_3(k^2t,\kappa'^2t), \quad (6)$$

with the scaling functions

$$g_0(k^2t,\kappa'^2t)=2\left(\frac{1}{\pi\tau'}\right)^{1/2}\frac{1}{k^2+\kappa'^2}\int_0^1 dy\,y^{-1/2}\left[\kappa'^2\exp\left(-\frac{\kappa'^2 t}{\tau'}y\right)+k^2\exp\left(-\frac{k^2+\kappa'^2}{\tau'}t+\frac{k^2 t}{\tau'}y\right)\right], \quad (7)$$

$$\begin{aligned}g_1(k^2t,\kappa'^2t)={}&\frac{\kappa b}{1+\kappa}\left(\frac{1}{\pi\tau'}\right)^{1/2}\left[-\pi\,{}_1F_1\left(\frac{1}{2};1;\frac{\kappa'^2 t}{\tau'}\right)\exp\left(-\frac{\kappa'^2 t}{\tau'}\right)+2\frac{k^2 t}{\tau'}\int_0^1 dy\,y^{-1/2}(1-y)^{1/2}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{k^2+\kappa'^2}{\tau'}t(1-y)\right)\right.\\&\left.\times\exp\left(-\frac{k^2+\kappa'^2}{\tau'}t+\frac{k^2 t}{\tau'}y\right)\right],\end{aligned} \quad (8)$$

$$g_2(k^2t,\kappa'^2t)=\frac{\kappa b}{1+\kappa}\,\frac{k^2+\kappa'^2}{\tau'}t\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{k^2+\kappa'^2}{\tau'}t\right)\exp\left(-\frac{k^2+\kappa'^2}{\tau'}t\right), \quad (9)$$

$$\begin{aligned}g_3(k^2t,\kappa'^2t)={}&\frac{\kappa b}{1+\kappa}\left(\frac{1}{\pi\tau'}\right)^{1/2}\frac{\kappa'^2 t}{2}\left[-\pi\,{}_1F_1\left(\frac{1}{2};1;\frac{\kappa'^2 t}{\tau'}\right)\exp\left(-\frac{\kappa'^2 t}{\tau'}\right)+2\frac{k^2 t}{\tau'}\int_0^1 dy\,y^{-1/2}(1-y)^{1/2}\right.\\&\left.\times\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{k^2+\kappa'^2}{\tau'}t(1-y)\right)\exp\left(-\frac{k^2+\kappa'^2}{\tau'}t+\frac{k^2 t}{\tau'}y\right)\right].\end{aligned} \quad (10)$$

Thus, from this scaling result of the structure factor it is shown that the domain size $L(t)$ scales as t^x and, correspondingly, the wall density $n(t)$ has the scaling form t^{-x}, which is consistent with the assumption made above before the calculation. Here the growth exponent x is 1/2 for the competing dynamics, the same as that for the (single) nonconserved Glauber system.

For $\kappa'^2 t \gg 1$, one can obtain the asymptotic result of $S(k,t)$, that is,

$$S(k,t) \to \frac{2\kappa'}{k^2+\kappa'^2} + \frac{\kappa b}{1+\kappa}\,\frac{1-\kappa'}{2}\,t^{-1/2}, \qquad (11)$$

while for $\kappa'^2 t \to 0$, the scaling result for quench to $T=0$, which is the main interest in the one-dimensional system, is given by

$$\begin{aligned} S(k,t) = {} & 2\left(\frac{t}{\pi\tau'}\right)^{1/2} \int_0^1 dy\, y^{-1/2} \exp\left[-\frac{k^2 t}{\tau'}(1-y)\right] \\ & + \frac{\kappa b}{1+\kappa}\left\{\left(\frac{1}{\pi\tau'}\right)^{1/2} \right. \\ & \times \left[-\pi + 2\frac{k^2 t}{\tau'}\int_0^1 dy\, y^{-1/2}(1-y)^{1/2} \right. \\ & \left. \times {}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{k^2 t}{\tau'}(1-y)\right)\exp\left(-\frac{k^2 t}{\tau'}(1-y)\right)\right] \\ & \left. + t^{-1/2}\frac{k^2 t}{\tau'}\,{}_1F_1\left(\frac{1}{2};\frac{3}{2};\frac{k^2 t}{\tau'}\right)\exp\left(-\frac{k^2 t}{\tau'}\right)\right\}, \end{aligned} \qquad (12)$$

which is of the scaling form

$$S(k,t) = t^{1/2} g_0(k^2 t) + g_1(k^2 t) + t^{-1/2} g_2(k^2 t). \qquad (13)$$

For $k^2 t \gg 1$, Eq. (12) becomes

$$S(k,t) \to 2(\pi\tau' t)^{-1/2} k^{-2} + \frac{\kappa b}{2(1+\kappa)} t^{-1/2}, \qquad (14)$$

which yields a correction to the Porod law [16].

It should be noted that the scaling forms of the structure factor (6) at nonzero temperature and Eq. (13) at zero temperature can be rewritten as

$$\begin{aligned} S(k,t) = {} & L g_0(kL,t/\tau) + g_1(kL,t/\tau) + L^{-1} g_2(kL,t/\tau) \\ & + L^{-2} g_3(kL,t/\tau), \end{aligned} \qquad (15)$$

with $\tau = 1/\kappa'^2 = \xi^2$ and $L \sim t^{1/2}$, and

$$S(k,t) = L g_0(kL) + g_1(kL) + L^{-1} g_2(kL), \qquad (16)$$

respectively. By Fourier transforming, the corresponding scaling of the equal-time pair correlation function is obtained

$$\begin{aligned} C(r,t) = {} & f_0(r/L,t/\tau) + L^{-1} f_1(r/L,t/\tau) + L^{-2} f_2(r/L,t/\tau) \\ & + L^{-3} f_3(r/L,t/\tau) \end{aligned} \qquad (17)$$

at nonzero temperature and

$$C(r,t) = f_0(r/L) + L^{-1} f_1(r/L) + L^{-2} f_2(r/L) \qquad (18)$$

at $T=0$. In the expressions of these scaling forms, the first leading term exhibits the scalings for the nonconserved dynamics in one dimension [9,12] associated with the spin-flip Glauber process, while the other terms are the corrections to scaling that arise from the spin-exchange Kawasaki process, which can be shown from the expressions of the scaling functions. In formulas (8)–(10) for quench to nonzero temperature, the correction functions g_1, g_2, and g_3 vary directly as $\kappa/(1+\kappa)$ and vanish if the contribution of the spin-exchange process is negligible or, equivalently, $\kappa \to 0$. Consequently, the scaling form $S(k,t) = t^{1/2} g_0(k^2 t, \kappa'^2 t)$ with the scaling function $g_0(x,y)$ and the asymptotic result $S(k,t) \to 2\kappa'/(k^2+\kappa'^2)$ for $\kappa'^2 t \gg 1$ obtained by Bray [12] for the one-dimensional Ising model with the (single) Glauber dynamics are recovered from Eqs. (6)–(11). Similar results can be obtained for zero temperature, that is, one can recover the well known scaling form $S(k,t) = t^{1/2} g_0(k^2 t)$ and the Porod law $S(k,t) \sim k^{-2}$ for Glauber Ising chain [12,13] from Eqs. (12)–(14). Moreover, from Eqs. (17) and (18) we note that the leading correction to the nonconserved scaling form of the pair correlation function is $L^{-1} f_1(r/L)$ here, originating from the spin-exchange process, while in the absence of this process, associated with departure of the initial condition from the scaling morphology, it is $L^{-4} f_1(r/L)$, according to the recent work of Bray *et al.* [10].

From the above scaling solution no correction to scaling of the domain size $L(t)$ and the wall density $n(t)$ is shown, as expected before the calculation. However, if assuming the correction to scaling of $L(t)$ and $n(t)$ before solving the evolution equation (3), e.g., $n(t) \sim b_1 t^{-x_1} + b_2 t^{-x_2} + \cdots$ with $0 < x_1 < x_2 < \cdots$, we find that the scaling solution similar to Eqs. (6) and (15) cannot be obtained.

In summary, we have shown that for this Ising system with competing Glauber and Kawasaki dynamics, the spin-exchange process with a configuration-independent rate has the effect of corrections to scaling, while the leading contribution to scaling is of the form for the nonconserved Glauber dynamics, as shown in Eqs. (5) and (12). More work is needed for a further understanding of these results.

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